45 YEARS OF EXCELLENCE

NARAYANA
COACHING CENTER
IIT- JEE/NEET/FOUNDATION

Time: 3 Hours | **Date: 08/09/2024**
Max Marks: 100 | **Number of Questions: 30**

IOQM SOLUTIONS (2024 – 25)

(Class – VIII to X)

Time : 180 Minutes | **Maximum Marks : 100**

INSTRUCTIONS TO CANDIDATES :

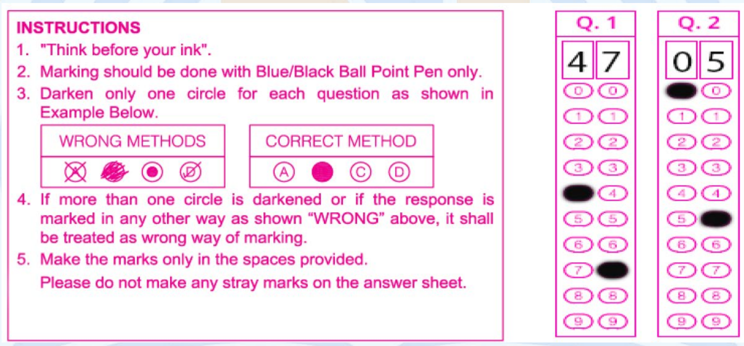
i) Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is STRICTLY PROHIBITED. Only ordinary pens and pencils are allowed inside the examination hall.

ii) The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a black or blue ball pen. Please DO NOT use a pencil or a gel pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

iii) The registration number and date of birth will be your login credentials for accessing your score.

iv) Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

v) Each question has a one- or two-digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.


INSTRUCTIONS
1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.

WRONG METHODS | CORRECT METHOD

4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
Please do not make any stray marks on the answer sheet.

vi) The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer. Questions 1 to 10 carry 2 marks each; questions 11 to 22 carry 5 marks each; questions 23 & 24 carry 10 marks each.

vii) All questions are compulsory.

viii) There are no negative marks.

ix) Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

x) After the exam, you may take away the Candidate's copy of the OMR sheet.

xi) Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.

xii) You may take away the question paper after the examination.

TO BE FILLED IN CAPITAL LETTERS

NAME OF THE STUDENT : ____________________

ROLL NUMBER : ☐☐☐☐ ☐☐☐☐ ☐☐☐☐☐☐

CLASS : ____________________

SCHOOL NAME : ____________________

Note :
1. gcd(a, b) denotes the greatest common divisor of integers a and b.
2. $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.
3. For a positive real number m, $\sqrt{m}$ denotes the positive square root of m. For example. $\sqrt{4} = +2$
4. Unless otherwise stated all numbers ae written in decimal notation.

Questions with Solutions

1. The smallest positive integer that does not divide 1 × 2 × 3 × 4 × 5 × 6 × 7 × 8 × 9 is

Ans. (11)

Sol. 1 × 2 × 3 × 4 × 5 × 6 × 7 × 8 × 9 × 9!

$9! = 2^7 \times 3^4 \times 5^1 \times 7^1$

∴ Smallest prime factor that not divide 91 is 11.

2. The number of four-digit odd numbers having digits 1,2,3,4 each occurring exactly once, is:

Ans. (12)

Sol.

_	_	_	_
↑	↑	↑	↑
3	2	1	2
ways	ways	ways	ways

∴ Total no. of ways = 3 × 2 × 1 × 2 = 12

3. The number obtained by taking the last two digits of 5^{2024} in the same order is

Ans. (25)

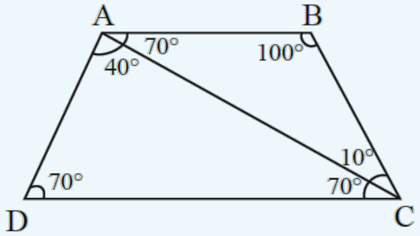
Sol. Last two digits of $5^{2024} = 5^{2024}$ mod 100 $\Rightarrow 5^2 = 25. 5^3 = 125, 5^4 = 625$

∴ for 5^n for n more than 2, last two digits are 25

4. Let ABCD be a quadrilateral with ∠ADC = 70°, ∠ACD = 70°, ∠ACD = 10° and ∠BAD = 110°. The measure of ∠CAB (in degrees) is

Ans. (70)

Sol.



$\angle D + \angle ACD + \angle DAC = 180^\circ$

$\angle DAC = 40^\circ$ So, $\angle CAB = 70^\circ$

$\therefore \angle CAB + \angle CAD = \angle DAB$

$\therefore y + 40^\circ = 110^\circ \quad \therefore y = 70^\circ$

5. Let $a = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, let $b = \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$ and let $c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$.

The value of $|ab - c|$ is

Ans. (1)

Sol. $a = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \qquad b = \frac{x}{z} + \frac{y}{x} + \frac{z}{y} \qquad c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$

$$ab = \frac{x^2}{yz} + 1 + \frac{xz}{y^2} + \frac{xy}{z^2} + \frac{xy}{x^2} + 1 + 1 + \frac{zy}{x^2} + \frac{z^2}{xy} = 3 + \frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy} + \frac{xz}{y^2} + \frac{xy}{z^2} + \frac{zy}{x^2}$$

$$= 3 + \frac{x^3 + y^3 + z^3}{xyz} + \frac{xz}{y^2} + \frac{xy}{z^2} + \frac{zy}{x^2}$$

$$c = \left(\frac{x}{z} + \frac{z}{y} + \frac{y^2}{z^2} + \frac{y}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right) = 2 + \frac{x^2}{yz} + \frac{z^2}{xy} + \frac{zx}{y^2} + \frac{y^2 z}{xz^2} + \frac{xy^2}{yz^2} + \frac{yz}{x^2}$$

$$|ab - c| = = 3 + \frac{x^3 + y^3 + z^2}{xyz} + \frac{xz}{y^2} + \frac{xy}{z^2} + \frac{zy}{x^2} - 2 - \frac{x^3}{xyz} - \frac{z^3}{xyz} - \frac{y^3}{xyz} - \frac{zx}{y^2} - \frac{xy}{x^2} - \frac{y^2}{x^2} = 1$$

6. Find the number of triples of real numbers (a, b, c) such that $a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$

Ans. (6)

Sol. $a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$

As $a^{20} \le a^{24}$

$\therefore$ even power $\Rightarrow$ a, b, c should be either 0, 1 or –1

If a = 1 or a = – 1 $\therefore a^{20}$ or $a^{24} = 1$

If b = 0 then $b^{24} = b^{20} = 0$ & c = 0

$\therefore$ Possible solutions are :

$a = \pm 1$,	$b = 0$,	$c = 0$
$a = 0$	$b = \pm 1$	$c = 0$
$a = 0$	$b = 0$	$c = \pm 1$

$\therefore$ Total No of Solutions 6

7. Determine the sum of all possible surface areas of a cube two of whose vertices are (1,2,0) and (3,3,2)

Ans. (99)

Sol. $A \equiv (1, 2, 0)$, $B \equiv (3,3,2)$

$AB = \sqrt{4+1+4} = 3$

AB will be either side, face diagram or main diagonal of cube.

So Case 1: AB is side $\rightarrow$ S.A. = $6\times(3)^2 = 54$

Case 2 : AB is main diagonal : $\sqrt{3}a = 3 \Rightarrow a = \sqrt{3}$

Surface area = 6 × 3 = 18

Case – 3 : AB face diagonal : $\sqrt{2}a = 3 \Rightarrow a = \frac{3}{\sqrt{2}}$

Surface are = $6 \times a^2 = 3 \times \frac{9}{2} = 27$

So, sum of different possible surface area = 54 + 18 + 27 = 99

8. Let n be the smallest integer such that the sum of digits of n is divisible by 5 as well as the sum of digits of (n + 1) is divisible by 5 . What are the first two digits of n in the same order?

Ans. (49)

Sol. n = 49999

$\therefore$ Sum = 4 + 9 + 9 + 9 + 9 = 40 divisible by 5

n = 50000 sum is 5 divisible by 5

So, 1st 2 digits of n is 49

9. Consider the grid of points $X = \{(m, n) | 0 \le m, n \le 4\}$.

We say a pair of points {(a, b), (c, d)}in X is a knight-move pair if ($c = a \pm 2$ and $d = b \pm 1$) or ($c = a \pm 1$ and $d = b \pm 2$). The number of knight-move pairs in X is:

Ans. (48)

Sol.

(0, –1) $\rightarrow$	(3, 1), (1, 2)	2
(1, +0) $\rightarrow$	(3, 1), (1, 2), (2, 0)	3
(2, 0) $\rightarrow$	(3, 2), (1, 2), (4, 1), (0, 1)	4
(3, –0) $\rightarrow$	(4, 2), (2, 2), (1, 0)	3
(4, 0) $\rightarrow$	(0, 1), (3, 2)	2

R_1 gives 14 ordered pairs.

R_2 gives 20 ordered pairs.

R_3 gives 28 ordered pairs.

R_4 gives 20 ordered pairs.

R^5 gives 14 ordered pairs.

= 96

So, total unordered pairs = 48

10. Determine the number of positive integral values of p for which there exists a triangle with sides a, b, and c which satisfy $a^2 + (p^2 + 9^2)\ b^2 + 9c^2) - 6ab - 6pbc = 0$

Ans. (5)

Sol. $a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = 0$

$a^2 - 6ab + 9b^2 + p^2b^2 + 9c^2 - 6pbc = 0$

$(a - 3b)^2 + (pb - 3c)^2 = 0 \Rightarrow a = 3b\ \&\ pb = 3c \Rightarrow \frac{a}{3} = b = \frac{c}{\left(\frac{p}{3}\right)} = \lambda$

Applying triangular inequality,

$3\lambda + \frac{p\lambda}{3} > \lambda \Rightarrow p > -6$

$3\lambda + \lambda > \frac{p\lambda}{3} \Rightarrow 12 > p$

$\frac{p\lambda}{3} + \lambda > 3\lambda \Rightarrow p > 6 \Rightarrow 6 < p < 12$

So, value of p is 05.

11. The positive real numbers a, b, c, satisfy :

$\frac{a}{2b+1} + \frac{2b}{3c+1} + \frac{3c}{a+1} = 1;$ $\frac{1}{a+1} + \frac{1}{2b+1} + \frac{1}{3c+1} = 2$

What is the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$?

Ans. (12)

Sol. Equ. (1) × (2)

$$\frac{2a}{2b+1}+\frac{4b}{3c+1}+\frac{6c}{a+1}=2 \quad \text{............ (A)}$$

$$\frac{1}{2b+1}+\frac{1}{3c+1}+\frac{1}{a+1} \quad \text{................ (B)}$$

Then, Equ. (A) – Equ. (B), we get

$$\frac{2a-1}{2b+1}+\frac{4b-1}{3c+1}+\frac{6c+1}{a+1}=0$$

As a, b, c > 0 = 2a – 1= 4b – 1 = 6c – 1 = 0 $\Rightarrow a=\frac{1}{2}, b=\frac{1}{4}, c=\frac{1}{6}$

$$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2+4+6=12$$

Alternative solutions

Let, a + 1 = x, 2b + 1 = y, 3c + 1 = z

$$\frac{x-1}{y}+\frac{y-1}{z}+\frac{z-1}{x}=1$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2 \Rightarrow \frac{x}{y}+\frac{y}{z}+\frac{z}{x}=3$$

But by AM $\geq$ GM (as x, y, z > 0)

$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq 3$ and equality holds iff x = y = z $\Rightarrow$ a + 1 = 2b + 1 = 3c + 1 $\Rightarrow$ a = 2b + 3c = $\frac{1}{2}$

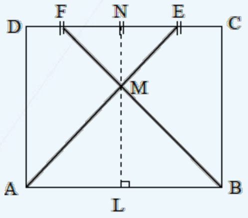
$$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{12}$$

12. Consider a square ABCD of side length 16. Let E, F be points on CD such that CE = EF = FD

Let the line BF and AE meet in M. The area of Δ MAB is

Ans. (96)

Sol.



The side of a sequence is 16

Then,

CF = EF = DE = $\frac{16}{3}$

Let MN = h,

Then, LM = 16 – h

Now, Δ MEF $\sim$ Δ MAB

$\therefore \frac{16-h}{h} = \frac{16}{\frac{16}{3}} \Rightarrow 16 - h = 3h \Rightarrow h = 4$

$\therefore (\Delta MAB) = \frac{1}{2} \times 8 \times (24) = 96 \text{sq.unit}$

13. Three positive integers a, b, c with a > c satisfies the following equations:

ac + b + c = bc + a + 66, a + b + c = 32 Find the value of a.

Ans. (19)

Sol: ac + b + c = bc + a + 66, a + b + c = 32

$\Rightarrow$ ac + b + c – bc – a = 66 $\Rightarrow$ c (a + 1) + b (1- c) – 1- a = 65

$\Rightarrow$ (c -1) (1 + a) + b (1 - c) = 65 = (c - 1) (1 + a - b) = 65

= (c - 1) (1 + a - b) = 1 × 65 × 5 × 13 = 13 × 5 = 65 × 1

	(c – 1)	(1 + a – b)	(a, b, c)	
Case – 1	1	65	(47, –17, 2)	Note possible as a, b, c are +ve integers
Case – 2	5	13	(19, 7, 6)	Possible as all given conditions are satisfied
Case – 3	13	5	(11, 7, 14)	Not possible as a > c
Case – 4	65	1	(–17, –17, 66)	Note possible as a, b are +ve integers

Only Possible value of a = 19

14. Initially, there are 3^{80} particles at the origin (0, 0). At each step the particles are moved to points above the x - axis as follows: if there are n particles at any point (x, y), then $\left\lfloor \frac{n}{3} \right\rfloor$ of them are moved to (x +1, y + 1) $\left\lfloor \frac{n}{3} \right\rfloor$ are and the remaining to (x-1, y+1).

For example, after the first step, there are 3^{79} particles each at (1,1), (0, 1) and (–1, 1). After the second step, there are 3^{78} particles each at (–2, 2) and (2, 2), 2×3^{78} particles each at (–1, 2) and (1, 2), and 3^{79} particles at (0, 2). After 80 steps, the number of particles at (79, 80) is:

Ans. (80)

Sol: $1 \rightarrow 3^{79}\ 3^{79}\ 3^{79}$

$2 \rightarrow 3^{78}, 2.3^{78}, 3.3^{78}, 2.3^{78}\ 3^{78}$

$3 \rightarrow 3^{77}, 3.3^{77}, 6.3^{77}, 7.3^{77}, 6.3^{77}, 3.3^{77}\ 3^{77}$

After 79th step (79,79) has 3 particles & (78,79) has 79 × 3 Particles

After 80th step (79,80) has $\frac{79\times3}{3}+\frac{3}{3}=80$

15. Let X be the set consisting of twenty positive integers n, n + 2, ……,n+38. The smallest value of n for which any three numbers a, b, $c \in X$, not necessarily distinct, form the sides of an acute-angled triangle is:

Ans. (92)

Sol: Sides: n, n, n + 38

$n + n > n + 38 \Rightarrow n > 38$

$\because$ It is acute

$n^2 + n^2 > (n + 38)^2 \Rightarrow n^2 - 76n > 38^2$

$n - 38 > \sqrt{2}.38$

$n > 91$

Therefore smallest n = 92

16. Let $f : R \rightarrow R$ be a function satisfying the relation 2 4 f (3– x) +3 f (x) = x^2 for any real x. Find the value of f (27)– f (25) to the nearest integer. (Here denotes the set of real numbers.)

Ans. (8)

Sol: $f : R \rightarrow R$ is a function such that

$4 f(3 - x) + 3f(x) = x^2$

Replace x by 3 – x,

Then 4 f(x) + 3 f(3 – x) = $(3 – x)^2$

On solving 1 and 2

$$f(x)=\frac{4(3-x)^2-3x^2}{7} \Rightarrow f(x)=\frac{x^2-24x+36}{7}$$

$$f(27)-f(25)=\frac{1}{7}\left[\left(27^2-25^2\right)-24(27-25)\right]=\frac{1}{7}[2\times 52-48] =\frac{56}{7}=8$$

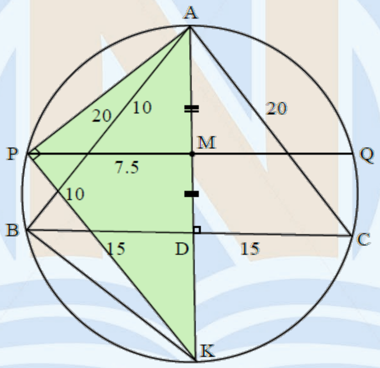
So, required value to nearest integer = 8

17. Consider an isosceles triangle ABC with sides BC = 30, CA = AB = 20. Let D be the foot of the perpendicular from A to BC, and let M be the midpoint of AD. Let PQ be chord of the circumcircle of triangle ABC, such that M lies on PQ and PQ is parallel to BC. The length of PQ is:

Ans. (25)

Sol. In right Δ PAK

PM^2 = AM.MK



$$=\frac{5\sqrt{7}}{2}\left(\frac{80\sqrt{7}}{7}-\frac{5\sqrt{7}}{2}\right)=\frac{5\sqrt{7}}{2}\times\sqrt{7}\times\frac{125}{14}=\frac{625}{2}$$

PM = $\frac{25}{2}$ = PQ = 2PM = 25

18. Let p, q be two-digit numbers neither of which are divisible by 10. Let r be the four-digit number by putting the digits of p followed by the digits of q (in order). As p, q vary, a computer prints r on the screen if gcd(p,q) = 1 and p + q divides r . Suppose that the largest number that is printed by the computer is N . Determine the number formed by the last two digits of N (in the same order).

Ans. (13)

Sol: $p + q \mid 100p + q \Rightarrow p + q \mid 99p$

As $(p, q) = 1 = (p + q, p) \Rightarrow p + q \mid 99$

$\therefore$ For largest value, $p + q = 99$.

q can't be 10.

If $q = 11 \Rightarrow p = 88$ but $(p, q) = 11$

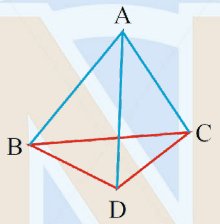
So $q = 12$ $p = 87$ but $(12, 87) = 3 \neq 1$

$\therefore$ $q = 13$ & $p = 86$ so $N = 8613$

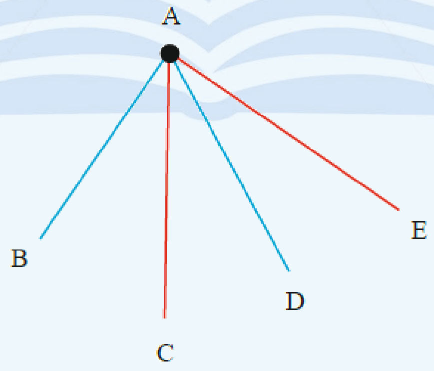
19. Consider five points in the plane, with no three of them collinear. Every pair of points among them is joined by a line. In how many ways can we color these lines by red or blue, so that no three of the points form a triangle with lines of the same color.

Ans. (12)

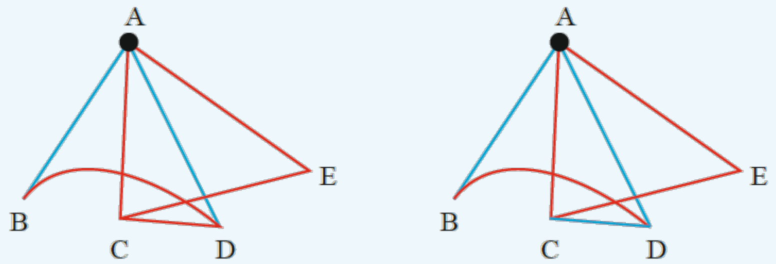
Sol: If from same vertex suppose we get 3 Blue say AC, AB, AD are all blue, then we can't color any of edge BC, BD, CD blue. So they all are red, and we get red Δ



Hence no 3 edges from any vertex are all of same colors, so each vertex has 2R &2B edges so there are ${}^4C_2$ ways to select colours of edges from the vertex A.



If AB, AD & AC, CE are red then, BD is red & CE is Blue and DC can be any colour R or B



Hence there are 6 × 2 = 12 ways

20. On a natural number n you are allowed two operations: (1) multiply n by 2 or (2) subtract 3 from n. For example, starting with 8 you can reach 13 as follows: $8 \to 16 \to 13$. You need two steps and you cannot do in less than two steps. Starting from 11, what is the least number of steps required to reach 121?

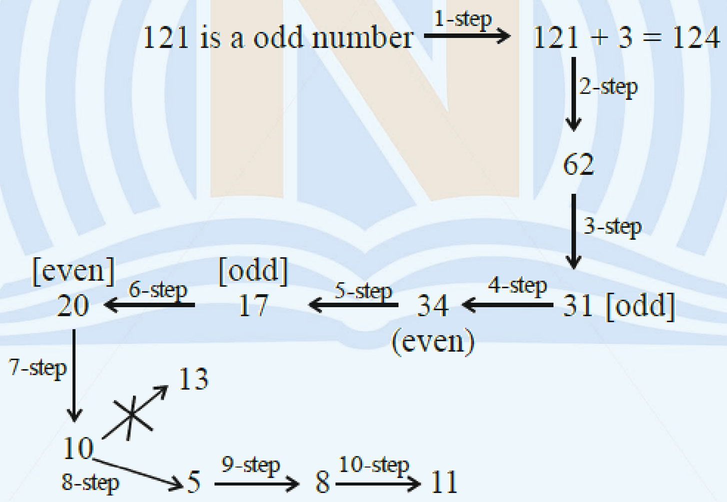
Ans. (10)

Sol: Reformulate the problem to reaching 11 starting from 121 by option:-

a) Divide n by 2

b) Add 3 to n.

Clearly, since 121 is a large number. Optimal solution would maximize the repetition of a).



21. An integer n is such that $\left\lfloor \frac{n}{9} \right\rfloor$ is a three digit number with equal digit, and $\left\lfloor \frac{n-172}{4} \right\rfloor$ is a 4 digit number with the digits 2, 0, 2, 4 in some order. What is the remainder when n is divided by 100 ?

Ans. (91)

Sol. $\left[\frac{n}{9}\right] = 3$ digit number with equal digits

$\Rightarrow \left[\frac{n}{9}\right] = (100+10+1)K = 111k \Rightarrow 111k \le \frac{n}{9} < 111k+1 \Rightarrow 999k \le n < +999k+9$

Where k = 1, 2, 3, 4, 5, ……7

n_{max} = 8991 + 9 = 9000 $\Rightarrow 2024 \le \left[\frac{n-172}{4}\right] \le 4220 \Rightarrow 2024 \le \frac{n-172}{4} < 4221$

$8096 \le n - 172 \le 16884$

$8268 \le n < 17056$

From (A) only possible values of n exist when k = 9.

For k = 9 $\Rightarrow$ 8991 $\le$ n < 8991 + 9

So n = 8991

When n is divided by 100 remainder is 91

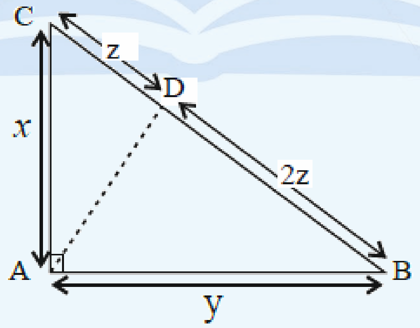
22. In a triangle ABC, $\angle BAC = 90^\circ$, Let D be the point on BC such that AB + BD = AC + CD.

Suppose BD : DC = 2 : 1.

If $\frac{AC}{AB} = \frac{m+\sqrt{p}}{n}$, where m, n are relatively prime positive integers and p is a prime number, determine the value of m + n + p.

Ans. (34)

Sol.



$x^2 + y^2 = 9z^2$

$x + z = y + 2z \Rightarrow x = y + z \Rightarrow x^2 + y^2 = 9(x - 4)^2$

$x^2 + y^2 = 9x^2 + 9y^2 - 18xy$

$4x^2 + 4y^2 - 9xy = 0 \Rightarrow 4\left(\frac{x}{4}\right)^2 - 9\left(\frac{x}{4}\right) + 4 = 0 \Rightarrow \frac{x}{y} = \frac{9 \pm \sqrt{81-64}}{8} = \frac{9+\sqrt{17}}{8}$ $\Rightarrow$ m + n + p

= 34

23. Consider the fourteen numbers, $1^4, 2^4, \ldots\ldots, 14^4$. The smallest natural number n such that they leave distinct remainders when divided by n is :

Ans: (31)

Sol : $1^4, 2^4, \ldots\ldots, 14^4$

$x^4 \equiv a \pmod n$

$y^4 \equiv b \pmod n$ such that $a \neq b$ for $x \neq y$ and $x, y \in \{1, 2, \ldots\ldots 14\}$

$(x^4 - y^4) = (a - b) \pmod n$

$\Rightarrow (x - y)(x + y)(x^2 + y^2) = (a - b) \pmod n \Rightarrow n \mid (x - y)(x + y)\, x^2 + y^2$ ....(i)

We have to find minimum n with condition (i)

Clearly, n > 27 as $(x + y) \in \{3, \ldots\ldots 27\}$

Now n = 28, x = 6, y = 8 works

n = 29, x = 5, y = 2 works

n = 30, x = 8, y = 2 works

for x = 31, there are no such x, y,

$$31 \mid \underbrace{(x-y)(x+y)(x^2+y^2)}_{\text{Must be prime tactor}}$$

$31 \mid (x^2 + y^2)$ and $31 \mid (x - y)(x + y) \Rightarrow 31$ will be the answer

24. Consider the set F of all polynomials whose coefficients are in the set of {0, 1}.

Let $q(x) = x^3 + x + 1$.

The number of polynomials p(x) in F of degree 14 such that the product p(x)q(x) is also in F is

Ans. (50)

Sol.. $p(x)q(x) = (x^{14} + \ldots.)(x^3 + x + 1)$

$P(x) = x^{14} \rightarrow 1$ case

$P(x) = x^{14} + x^{\alpha}$

Where $\alpha = 10, 9, 8, \ldots\ldots 1, 0 \rightarrow 11$ case

$\alpha = 10,$ $\alpha = 9,$ $\alpha = 8,$ $\alpha = 7,$ $\alpha = 6,$ $\alpha = 5,$ $\alpha = 4,$	$\beta = 6, 5, 4, 3, 2, 1, 0$ $\beta = 5, 4, 3, 2, 1, 0$ $\beta = 4, 3, 2, 1, 0$ $\beta = 3, 2, 1, 0$ $\beta = 2, 1, 0$ $\beta = 1, 0$ $\beta = 0$	}	$\rightarrow$ 28 cases

$p(x) = x^{14} + x^{\alpha} + x^{\beta} + x^{\gamma}$

$\alpha = 10, \quad \beta = 6\ \gamma = 2, 1, 0$
$\alpha = 10, \quad \beta = 5\ \gamma = 1, 0$
$\alpha = 10, \quad \beta = 4\ \gamma = 0$
} $\rightarrow$ 6 cases

$\alpha = 9, \quad \beta = 5\ \gamma = 1, 0$
$\beta = 4\ \gamma = 0$
} $\rightarrow$ 3 cases

$\alpha = 8, \ \beta = 4, \ \gamma = 0\} \rightarrow 1$ cases

Hence, total case = 1 + 11 + 28 + 6 + 3 + 1 = 50 cases

25. A finite set M of positive integers consists of distinct perfect squares and the number 92. The average of the numbers in M is 85. If we remove 92 from M, the average drops to 84. If N^2 is the largest possible square in M, what is the value of N ?

Ans. (22)

Sol. $M = \{a_1^2, a_2^2, ...a_k^2, 92\}$

$a_1^2, a_2^2, ...a_k^2 + 92 = 85(k+1)$

$a_1^2, a_2^2, ...a_k^2 = 84k \Rightarrow 92 = 85k + 85 - 84k \Rightarrow k = 7$

$84 \times 7 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 7^2 + 22^2 = 588$

N = 22

26. The sum of $\lfloor x \rfloor$ for all real numbers x satisfying the equation $16 + 15x + 15x^2 = \lfloor x \rfloor^3$ is

Ans. (33)

Sol. As $\quad x - 1 < [x] \le x$

So $x^3 \ge 15x^2 + 15x + 16$ gives $x \ge 16$

So possible value of x are 16, 17

$\therefore$ sum of x = 33

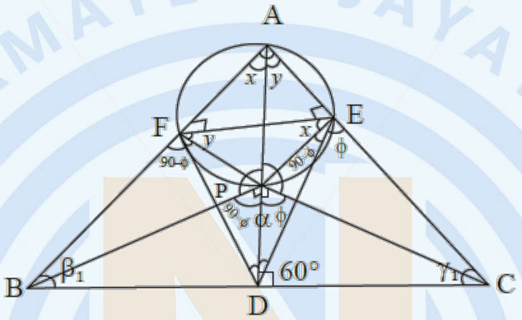
27. In a triangle ABC, a point P in the interior of ABC is such that $\angle BPC - \angle BAC = \angle CPA - \angle CBA$

$= \angle APB - \angle ACB$

Suppose $\angle BAC = 30^\circ$ and AP = 12. Let D, E, F be the feet of perpendiculars form P on to BC, CA, AB respectively. If $m\sqrt{n}$ is the area of the triangle DEF where m, n are integers with n prime, then what is the value of the product mn ?

Ans. (27)

Sol.



$\alpha - \alpha_1 = \beta - \beta_1 = \gamma - \gamma_1 = \theta$ (Say), Given $\alpha_1 = \frac{\pi}{6}$

Observe AFPE is cyclic $\Rightarrow \frac{\sin\alpha_1}{EF} = \frac{1}{AP} \Rightarrow EF = AP \sin\alpha_1 = 12 \times \frac{1}{2} = 6$

$\alpha + \beta + \gamma = \alpha_1 + \beta_1 + \gamma_1 + 3\theta$

$\left.2\pi = \pi + 3\theta \Rightarrow \theta = \frac{\pi}{3}\right\} \Rightarrow \alpha = \alpha_1 + \frac{\pi}{3} = \frac{\pi}{2}$

By angle change we get $\angle FDE = 60^\circ$

If the area is constant irrespective of the position of P then for AP = 12 (P moving on circle with center A) a some position of P, DF = ED $\Rightarrow \Delta$DEF to be equilateral

Area $= \frac{\sqrt{3}}{4}(EF)^2 = 9\sqrt{3} \Rightarrow mn = 27$

28. Find the largest positive integer n < 30 such that $\frac{1}{2}(n^8 + 3n^4 - 4)$ is not divisible by the square of any prime number

Ans. (20)

Sol. If n is odd $n^4 \equiv 1 \pmod 8$

$\frac{1}{2}(n^8 + 3n^4 - 4) \equiv 0 \pmod 4$

So we want n even

If $x^2 + 3x - 4 = (x^2 + 4)(x^2 - 1) \equiv 0 \pmod{b^2}$

$\phi^2 | \frac{1}{2}(n^8 + 3n^4 - 4) = \frac{1}{2}(n^4 + 4)(n^4 - 1) = \frac{1}{2}(n^4 + 4)(n^2 + 1)(n + 1)(n - 1)$

$= \frac{1}{2}(n^4 + 4)(n^2 + 1)(n + 1)(n - 1) = \frac{1}{2}((n+1)^2 + 1)((n-1)^2 + 1)(n + 1)(n - 1)(n^2 + 1)$

By trial and error

n = 28 $3^2 | 27 = 28 - 1$

n = 26 $5^2 | 26 - 1$

n = 24 $5^2 | 24 + 1$

n = 22 $\text{Exp} = \frac{1}{2}(530)(484 + 1)(21)(23)$

5^2 | Exp.

n = 20 $\text{Exp} = \frac{1}{2}(442)(362)(21)(19)(401)$

works

29. Let $n = 29^{19}\, 3^{12}$.

Let M denote the number of positive divisors of n^2 which are less than n but would not divide n. What is the number formed by taking the last two digits of M (n the same order) ?

Ans. (28)

Sol. The divisor of n^2 that does not divide n and is less than n will be of type $2^a \cdot 3^b$ where $a \le 38$, $b \le 24$

Now in order that if does not divide n

Either a > 19 or b > 12

Consider divisor of type $2^{19+x}, 3^{12-y}$

$= n \times \frac{2^x}{3^y} > n$ if $2^x > 3^y$

$< n$ if $2^x < 3^y$

For each (x, y), $1 \le x \le 19$ & $1 \le y \le 12$

There is unique required factor of n^2 which is less then n & does not divide n.

No of such factor = 12 × 19 = 228

Ans = 28

30. Let ABC be a right-angled triangled with $\angle B = 90°$. Let the length of the altitude BD be equal to 12. What is the minimum possible length of AC, given that AC and the perimeter of triangle ABC are integers ?

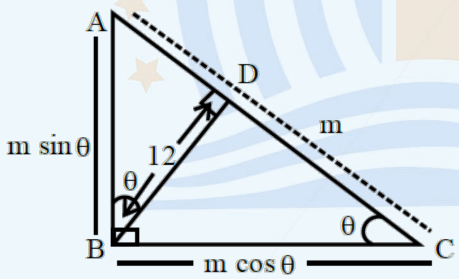
Ans. (25)

Sol. Let AC = m and perimeter is m + n

$\Rightarrow AB + BC = n$

$AB = m\sin\theta$, $BC = m\cos\theta \Rightarrow m(\sin\theta + \cos\theta) = n$

$m \sin\theta\cos\theta = 12 \Rightarrow m > 24$



$\Rightarrow m^2\left(1 + \frac{2 \times 12}{m}\right) = n^2 \Rightarrow \frac{m^2(m+24)}{m} = n^2 \Rightarrow m(m+24) = n^2$

$m = 25$ & $n^2 = 49 \times 25$

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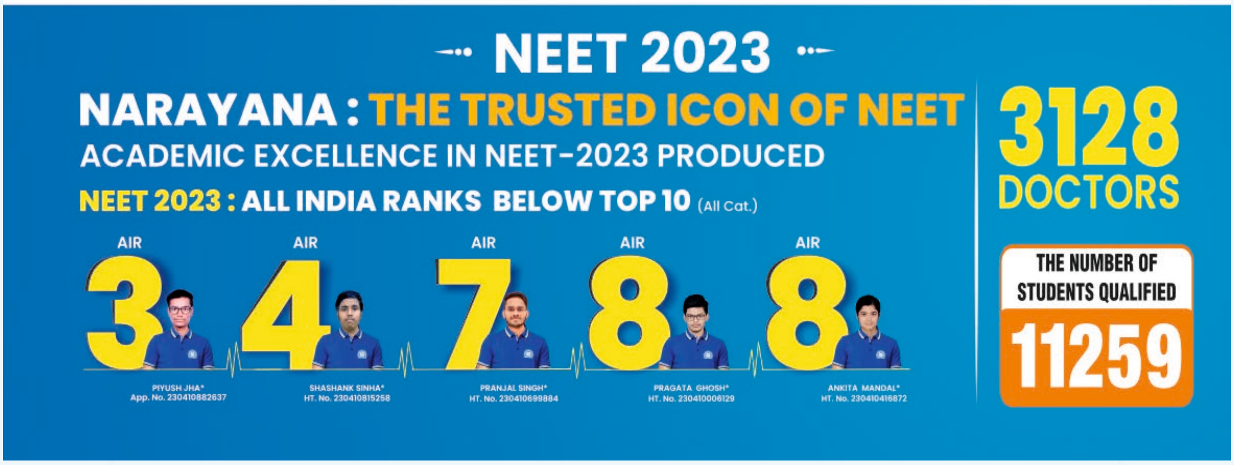
ARYAN PRAKASH
12 AIR
ROHAN SAI PABBA
6 in Top 10 Open Category
7 in Top 12 Open Category
28 in Top 100 Open Category
891 in Top 1000 All Category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